EMBEDDING THE FLAG REPRESENTATION IN DIVIDED POWERS

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Abstract. A generalization of a theorem of Crabb and Hubbuck concerning the embedding of flag representations in divided powers is given, working over an arbitrary finite field $\mathbb{F}$, using the category of functors from finite-dimensional $\mathbb{F}$-vector spaces to $\mathbb{F}$-vector spaces.

1. Introduction

Let V be a finite-dimensional $\mathbb{F}$-vector space over a finite field $\mathbb{F}$; the flag variety of complete flags of length r in V induces a permutation representation $\mathbb{F}[\mathfrak{Flag}_r](V)$ of the general linear group $GL(V)$, which is of interest in representation theory. The notation is derived from the fact that the flag representation arises as the evaluation on the space V of a functor $\mathbb{F}[\mathfrak{Flag}_r]$ in the category $\mathscr{F}$ of functors from finite-dimensional $\mathbb{F}$-vector spaces to $\mathbb{F}$-vector spaces; similarly, the divided power functors Γ^n induce $GL(V)$-representations $\Gamma^n(V)$. Motivated by questions from algebraic topology (and working over the prime field $\mathbb{F}_2$), Crabb and Hubbuck [1, 3] associated to a sequence $\underline{s}$ of integers, $s_1 \geq \ldots \geq s_r > s_{r+1} = 0$, a morphism

$$\mathbb{F}_2[\mathfrak{Flag}_r](V) \rightarrow \Gamma^{[\underline{s}]_2}(V) \tag{1}$$

or $GL(V)$-modules, where $[\underline{s}]_q$ is the integer $\Sigma_{i=1}^r(q^{s_i} - 1)$. The motivating observation of this paper is that this is defined globally as a natural transformation

$$\phi_{\underline{s}} : \mathbb{F}[\mathfrak{Flag}_r] \rightarrow \Gamma^{[\underline{s}]_q} \tag{2}$$

in the functor category $\mathscr{F}$ and for each finite field $\mathbb{F}$, where $q = |\mathbb{F}|$.

The Crabb-Hubbuck morphism $\phi_{\underline{s}}$ arises in the construction of the ring of lines (developed independently in the dual situation by Repka and Selick [8]). Namely, the divided power functors form a commutative graded algebra in $\mathscr{F}$ and the ring of lines is the graded sub-functor generated by the images of the morphisms $\phi_{\underline{s}}$, which forms a sub-algebra of Γ^*.

The ring of lines is of interest in relation to the study of the primitives under the action of the Steenrod algebra on the singular homology $H_*(BV;\mathbb{F}_2)$ of the classifying space of V; the primitives arise in a number of questions in algebraic topology. This relation can be explained from the point of view of the functor category $\mathscr{F}$ as follows, by identifying $H_*(BV;\mathbb{F}_2)$ with the graded vector space $\Gamma^*(V)$. Steenrod reduced power operations correspond to natural transformations of the form $\Gamma^a \rightarrow \Gamma^b$, $a \geq b$ (see [6]). Define the Steenrod kernel functors K^a by:

$$K^a := \mathrm{Ker}\{\Gamma^a \rightarrow \bigoplus_{f \in \mathrm{Hom}(\Gamma^a,\Gamma^b), a>b} \Gamma^b\}.$$

These functors form a commutative graded algebra in $\mathscr{F}$. The primitives in $H_*(BV;\mathbb{F}_2)$ are obtained by evaluating on V. The analysis of the primitives is dual to the study of the indecomposables for the action of the Steenrod reduced powers on $H^*(BV;\mathbb{F}_2)$; this is a difficult problem which has attracted much interest. For the

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field $\mathbb{F}_2$, the complete structure is known only for spaces of dimension at most four; the case of dimension three is due to Kameko and a published account is available in Boardman [5, 2]; Kameko also announced the case of dimension four, which has since been calculated by Sum, a student of Nguyen Hung.

The morphism $\phi_{\underline{s}}$ of equation (2) maps to $K^{[\underline{s}]_2}$ for elementary reasons and the ring of lines is a graded sub-algebra in $\mathscr{F}$ of K^*. A fundamental question is to determine in which degrees the ring of lines coincides with the Steenrod kernel. Motivated by this question, Crabb and Hubbuck [3, Proposition 3.10] gave an explicit criterion upon the sequence (s_i) with respect to the dimension of V for the morphism $\phi_{\underline{s}}$ to be a monomorphism.

The purpose of this note is two-fold; to present a proof exploiting the category $\mathscr{F}$ and to generalize the result to an arbitrary finite field $\mathbb{F}$, with $q = |\mathbb{F}|$. The main result of the paper is the following:

Theorem 1. *Let r be a natural number and $\underline{s} = (s_1 > \ldots > s_r > s_{r+1} = 0)$ be a sequence of integers which satisfies the condition $[s_i - s_{i+1}]_q \geq (q-1)(\dim V - i + 1)$, for $1 \leq i \leq r$; then the morphism $\phi_{\underline{s}}$ induces a monomorphism*

$$\mathbb{F}[\mathfrak{Flag}_r](V) \hookrightarrow \Gamma^{[\underline{s}]_q}(V).$$

The proof sheds light upon the method proposed by Crabb and Hubbuck; namely, the proof establishes the stronger result that the composite with a morphism induced by the iterated diagonal on divided power algebras and the Verschiebung morphism is a monomorphism. This is of interest in the light of recent work over the field $\mathbb{F}_2$ by Grant Walker, Reg Wood [9] and Tran Ngoc Nam generalizing the result of Crabb and Hubbuck, relaxing the required hypothesis on the sequence (s_i).

These techniques can be used to provide further information on the nature of the embedding results; for instance:

Proposition 2. *Let $\underline{s}$ be a sequence of integers $(s_1 > \ldots > s_r > s_{r+1} = 0)$ and V be a finite-dimensional vector space for which the morphism $\phi_{\underline{s}}(V) : \mathbb{F}[\mathfrak{Flag}_r](V) \rightarrow \Gamma^{[\underline{s}]_q}(V)$ is a monomorphism.*

Let $\underline{s^+}$ denote the sequence given by $s_i^+ = s_i + 1$, for $1 \leq i \leq r$, and $s_{r+1}^+ = 0$. Then the morphism

$$\phi_{\underline{s^+}}(V) : \mathbb{F}[\mathfrak{Flag}_r](V) \rightarrow \Gamma^{[\underline{s^+}]_q}(V)$$

is a monomorphism.

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Contents

2. Preliminaries

Fix a finite field $\mathbb{F} = \mathbb{F}_q$, and write $q = p^m$, where p is the characteristic of $\mathbb{F}$. Let $\mathscr{F}$ be the category of functors from finite-dimensional $\mathbb{F}$-vector spaces to $\mathbb{F}$-vector

spaces. The group of units $\mathbb{F}^\times$ is isomorphic to the cyclic group $\mathbb{Z}/(q-1)\mathbb{Z}$ via $i \mapsto \lambda^i$, for a generator λ. In particular, the group has order prime to p, hence the category of $\mathbb{F}[\mathbb{F}^\times]$-modules is semi-simple. This gives rise to the weight splitting of $\mathscr{F}$ (as in [6]):

$$\mathscr{F} \cong \prod_{i \in \mathbb{Z}/(q-1)\mathbb{Z}} \mathscr{F}^i,$$

where $\mathscr{F}^i$ is the full subcategory of functors such that $F(\lambda 1_V) = \lambda^i F(1_V)$ for all finite-dimensional spaces V and $\lambda \in \mathbb{F}^\times$. By reduction mod $q-1$, the weight category $\mathscr{F}^k$ can be taken to be defined for k an integer.

The duality functor $D : \mathscr{F}^{\mathrm{op}} \to \mathscr{F}$ is defined by $DF(V) := F(V^*)^*$ and D restricts to a functor $D : (\mathscr{F}^k)^{\mathrm{op}} \to \mathscr{F}^k$

2.1. **Divided powers and the Verschiebung.** Recall that Γ^k denotes the kth divided power functor, defined as the invariants $\Gamma^k := (T^k)^{\mathfrak{S}_k}$ under the action of the symmetric group permuting the factors of T^k, the kth tensor power functor. (By convention, the divided power Γ^0 functor is the constant functor $\mathbb{F}$ and $\Gamma^i = 0$ for $i < 0$). The divided power functor Γ^k is dual to the kth symmetric power functor S^k and the functors Γ^k, T^k, S^k all belong to the weight category $\mathscr{F}^k$.

The divided power functors Γ^* form a graded exponential functor; namely for finite-dimensional vector spaces U, V and a natural number n, there is a binatural isomorphism

$$\Gamma^n(U \oplus V) \cong \bigoplus_{i+j=n} \Gamma^i(U) \otimes \Gamma^j(V).$$

This has important consequences (see [4], for example); in particular, for pairs of natural numbers (a, b), there are cocommutative coproduct morphisms $\Gamma^{a+b} \xrightarrow{\Delta} \Gamma^a \otimes \Gamma^b$ and commutative product morphisms $\Gamma^a \otimes \Gamma^b \xrightarrow{\mu} \Gamma^{a+b}$, which are coassociative (respectively associative) in the appropriate graded sense.

The Verschiebung is a natural surjection $\mathcal{V} : \Gamma^{qn} \twoheadrightarrow \Gamma^n$, for integers $n \geq 0$, dual to the Frobenius qth power map on symmetric powers. More generally there is a Verschiebung morphism $\mathcal{V}_p : \Gamma^{np} \twoheadrightarrow \Gamma^{(1)}$, dual to the Frobenius pth power map, where $(-)^{(1)}$ denotes the Frobenius twist functor (see [4]). The Verschiebung $\mathcal{V}$ is obtained by iterating $\mathcal{V}_p$ m times, where $q = p^m$.

The pth truncated symmetric power functor $\overline{S}^n$ is defined by imposing the relation $v^p = 0$; similarly the qth truncated symmetric power functor $\tilde{S}^n$ is given by forming the quotient by the relation $v^q = 0$. There is a natural surjection $\tilde{S}^n \twoheadrightarrow \overline{S}^n$; over a prime field the functors coincide.

Dualizing gives the following:

Definition 2.1. For n a natural number,

(1) let $\tilde{\Gamma}^n$ denote the kernel of the composite

$$\Gamma^n \xrightarrow{\Delta} \Gamma^{n-q} \otimes \Gamma^q \xrightarrow{1 \otimes \mathcal{V}} \Gamma^{n-q} \otimes \Gamma^1;$$

(2) let $\overline{\Gamma}^n$ be the kernel of the composite

$$\Gamma^n \xrightarrow{\Delta} \Gamma^{n-p} \otimes \Gamma^p \xrightarrow{1 \otimes \mathcal{V}_p} \Gamma^{n-p} \otimes (\Gamma^1)^{(1)}.$$

Lemma 2.2. *Let n be a natural number.*

(1) *The functor $\tilde{\Gamma}^n$ is dual to $\tilde{S}^n$.*
(2) *The functor $\overline{\Gamma}^n$ is isomorphic to $\overline{S}^n$ and is simple.*
(3) $\tilde{\Gamma}^n(\mathbb{F}^d) = 0$ *if* $d \leq (n-1)/(q-1)$.

Proof. The first statement follows from the definitions and the identification $DS^n = \Gamma^n$. The simplicity of $\overline{S}^n$ is a standard fact [6]; the isomorphism follows from the fact that the simple functors of $\mathscr{F}$ are self-dual [7].

The final statement is an elementary verification, which is a consequence of the observation that the maximal degree of a free q-truncated symmetric algebra on d variables of degree one is $d(q-1)$. $\square$

2.2. **Further properties of divided powers.**

Notation 2.3. For s a natural number, let $[s]_q$ denote the integer $q^s - 1$ and, for $\underline{s}$ a sequence of integers, $s_1 \geq \ldots \geq s_r > s_{r+1} = 0$, let $[\underline{s}]_q$ denote $\Sigma_i [s_i]_q$.

Notation 2.4. The element of $\Gamma^k(V)$ corresponding to the symmetric tensor $x^{\otimes k} \in T^k(V)$, for x an element of V, will be denoted simply by $x^{\otimes k}$.

Lemma 2.5. *Let s be a positive integer. For any $x \in V$, the class $x^{\otimes [s]_q} \in \Gamma^{[s]_q}(V)$ is equal to the product*

$$\prod_{j=0}^{sm-1} x^{\otimes (p-1)p^j},$$

where $|\mathbb{F}| = q = p^m$.

This Lemma is a consequence of the following well-known general property of the divided power functors.

Lemma 2.6. *Let $a_0, \ldots, a_t$ and $0 = r_0 < r_1 < \ldots < r_t$ be sequences of natural numbers such that, for $0 \leq i < t$, $a_i p^{r_i} < p^{r_{i+1}}$. Then the composite*

$$\Gamma^{\Sigma_{i=0}^t a_i p^{r_i}} \to \bigotimes_{i=0}^{t} \Gamma^{a_i p^{r_i}} \to \Gamma^{\Sigma_{i=0}^t a_i p^{r_i}}$$

is an isomorphism, where the first morphism is the coproduct and the second the product.

Proof. Using the (co)associativity of the product (respectively the coproduct), it suffices to prove the result for $t = 1$. The composite morphism in this case is multiplication by the scalar $\binom{a_0 + a_1 p^{r_1}}{a_0}$, which is equal to one modulo p, since $a_0 < p^{r_1}$, by hypothesis. $\square$

The following Lemma is the key to the construction of the Crabb-Hubbuck morphism, $\phi_{\underline{s}}$, in Section 4.

Lemma 2.7. *Let x be an element of V, s be a natural number and $0 < i \leq [s]_q$ be an integer. The product $x^{\otimes [s]_q} x^{\otimes i}$ is zero in $\Gamma^{[s]_q + i}(V)$.*

Proof. The element $x^{\otimes [s]_q}$ is equal to the product $\prod_{j=0}^{sm-1} x^{\otimes (p-1)p^j}$, by Lemma 2.5. Similarly, considering the p-adic expansion $i = \Sigma_{j=0}^{sm-1} i_j p^j$, where $0 \leq i_j < p$ and at least one i_j is non-zero, there is an equality $x^{\otimes i} = \prod_{j=0}^{sm-1} x^{\otimes i_j p^j}$. The product is associative and commutative, hence it suffices to show that, if $i_j \neq 0$, then $x^{\otimes (p-1)p^j} x^{\otimes i_j p^j}$ is zero. Up to non-zero scalar in $\mathbb{F}_p^\times$, the element $x^{\otimes i_j p^j}$ is the i_j-fold product of $x^{\otimes p^j}$ (since $0 < i_j < p$, by hypothesis), hence it suffices to show that $x^{\otimes (p-1)p^j} x^{\otimes p^j}$ is zero. This element identifies with $\binom{p^{j+1}}{p^j} x^{\otimes p^{j+1}}$ and the scalar is zero in $\mathbb{F}$.

$\square$

The behaviour of the Verschiebung morphism with respect to products is important.

Lemma 2.8. *Let $\beta_1, \ldots, \beta_k$ be positive integers such that $\Sigma_{i=1}^k \beta_i = pN$ for some integer N. The composite*

$$\bigotimes_{i=1}^{k} \Gamma^{\beta_i} \xrightarrow{\mu} \Gamma^{pN} \xrightarrow{\mathcal{V}_p} (\Gamma^N)^{(1)},$$

where $\mathcal{V}_p$ is the Verschiebung and μ is the product, is trivial unless for each i, $\beta_i = p\beta'_i$, $\beta'_i \in \mathbb{N}$. In this case, there is a commutative diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^k \Gamma^{p\beta'_i} \arrow[r, "\mu"] \arrow[d, "\bigotimes \mathcal{V}_p"'] & \Gamma^{pN} \arrow[d, "\mathcal{V}_p"] \\
\bigotimes_{i=1}^k (\Gamma^{\beta'_i})^{(1)} \arrow[r, "\mu^{(1)}"'] & (\Gamma^N)^{(1)}.
\end{tikzcd}$$

Proof. The statement is more familiar in the dual situation, where it corresponds to the fact that the diagonal of the symmetric power algebra commutes with the Frobenius. □

Remark 2.9. An analogous statement holds for iterates of $\mathcal{V}_p$ and for the Verschiebung $\mathcal{V} : \Gamma^{qN} \to \Gamma^N$.

3. Projectives and flags

This section introduces the flag functors and relates them to the standard projective generators of the category $\mathscr{F}$.

3.1. **The projective and flag functors.** For r a natural number, the standard projective object $P_{\mathbb{F}^r}$ in $\mathscr{F}$ is given by $P_{\mathbb{F}^r}(V) = \mathbb{F}[\mathrm{Hom}(\mathbb{F}^r, V)]$ and is determined up to isomorphism by $\mathrm{Hom}_{\mathscr{F}}(P_{\mathbb{F}^r}, G) = G(\mathbb{F}^r)$. In particular, the projective $P_{\mathbb{F}}$ is the functor $V \mapsto \mathbb{F}[V]$.

The weight splitting determines a direct sum decomposition

$$P_{\mathbb{F}} \cong \bigoplus_{i \in \mathbb{Z}/(q-1)\mathbb{Z}} P^i_{\mathbb{F}}$$

in which $P^i_{\mathbb{F}}$ is indecomposable for $i \neq 0$ and $P^0_{\mathbb{F}}$ admits a decomposition $P^0_{\mathbb{F}} = \mathbb{F} \oplus \overline{P^0_{\mathbb{F}}}$ (Cf [6, Lemma 5.3], noting that Kuhn uses a splitting associated to the multiplicative semigroup $\mathbb{F}$).

There is a Knneth isomorphism for projectives so that, for r a positive integer, $P_{\mathbb{F}}^{\otimes r}$ is projective and identifies with the projective functor.

Definition 3.1. For r a positive integer, let $\mathbb{F}[\mathfrak{Flag}_r]$ be the functor which is defined in terms of complete flags of length r as follows.

As a vector space, $\mathbb{F}[\mathfrak{Flag}_r](V)$ has basis the set of complete flags of length r. A morphism $V \to W$ sends a complete flag to its image, if this is a complete flag, and to zero otherwise.

Recall that a functor F of $\mathscr{F}$ is said to be constant-free if $F(0) = 0$.

Lemma 3.2. *Let $r \geq s > 0$ be integers.*

1. *The functor $\mathbb{F}[\mathfrak{Flag}_r]$ is constant-free and belongs to $\mathscr{F}^0$.*
2. *There is a diagonal morphism in $\mathscr{F}$:*

$$\mathbb{F}[\mathfrak{Flag}_r] \to \mathbb{F}[\mathfrak{Flag}_r] \otimes \mathbb{F}[\mathfrak{Flag}_r].$$

3. *There is a surjection*

$$\pi_{r,s} : \mathbb{F}[\mathfrak{Flag}_r] \twoheadrightarrow \mathbb{F}[\mathfrak{Flag}_s]$$

which forgets the subspaces of dimension greater than s.

3.2. **Structure of the projectives.**

Lemma 3.3. *Let $n \geq 1$ be an integer, then* $\mathrm{Hom}_{\mathscr{F}}(\overline{P^0_{\mathbb{F}}}, \Gamma^{n(q-1)}) = \mathbb{F}$.

Proof. The result follows from the Yoneda lemma, the weight splitting and the fact that $n \geq 1$ allows passage to the constant-free part, $\overline{P^0_{\mathbb{F}}}$. □

Recall that a functor is said to be finite if it has a finite composition series.

Proposition 3.4. [6, 7]

(1) *The surjection $P_{\mathbb{F}} \twoheadrightarrow \mathbb{F}[\mathfrak{Flag}_1]$ induces an isomorphism $\overline{P^0_{\mathbb{F}}} \cong \mathbb{F}[\mathfrak{Flag}_1]$.*
(2) *There is an inverse system of finite functors $\ldots \to \mathfrak{q}_k\overline{P^0_{\mathbb{F}}} \to \mathfrak{q}_{k-1}\overline{P^0_{\mathbb{F}}} \to \ldots$ such that*
  (a) $\overline{P^0_{\mathbb{F}}} \cong \lim_{\leftarrow} \mathfrak{q}_k\overline{P^0_{\mathbb{F}}}$;
  (b) *for $k \geq 1$, $\mathfrak{q}_k\overline{P^0_{\mathbb{F}}}$ is isomorphic to the image of any non-trivial morphism $\overline{P^0_{\mathbb{F}}} \to \Gamma^{k(q-1)}$;*
  (c) *for $k \geq 2$, there is a non-split short exact sequence*
$$0 \to \tilde{\Gamma}^{k(q-1)} \to \mathfrak{q}_k\overline{P^0_{\mathbb{F}}} \to \mathfrak{q}_{k-1}\overline{P^0_{\mathbb{F}}} \to 0.$$
(3) *The functor $\overline{P^0_{\mathbb{F}}}$ is dual to a locally-finite functor.*
(4) *If $\mathbb{F}$ is the prime field $\mathbb{F}_p$, then the functor $\overline{P^0_{\mathbb{F}}}$ is uniserial with composition factors $\{\overline{\Gamma}^{k(p-1)} | k \geq 1\}$, each occurring with multiplicity one.*

Proof. It is more straightforward to deduce the result from the description of the dual $D\overline{P^0_{\mathbb{F}}}$; this is isomorphic to the functor $(\bigoplus_{k=0}^{\infty} S^{k(q-1)})/\langle v^q - v\rangle$, where the relation is induced by the weight zero part of the ideal $\langle v^q - v\rangle$ (this is deduced from [6, Lemma 4.12] by applying the evident weight splitting). □

It is important to have a measure of how good an approximation $\mathfrak{q}_k\overline{P^0_{\mathbb{F}}}$ is to $\overline{P^0_{\mathbb{F}}}$.

Lemma 3.5. *Let $k \geq 1$ be an integer. Up to scalar in $\mathbb{F}^\times$, there is a unique non-trivial morphism $\overline{P^0_{\mathbb{F}}} \to \Gamma^{k(q-1)}$. Any non-trivial morphism $\overline{P^0_{\mathbb{F}}} \to \Gamma^{k(q-1)}$*

(1) *factors as*
$$\overline{P^0_{\mathbb{F}}} \twoheadrightarrow \mathfrak{q}_k\overline{P^0_{\mathbb{F}}} \hookrightarrow \Gamma^{k(q-1)}$$
and
(2) *induces a monomorphism*
$$\overline{P^0_{\mathbb{F}}}(V) \hookrightarrow \Gamma^{k(q-1)}(V)$$
if $\dim V \leq k$.

Proof. The unicity follows from Lemma 3.3 and the factorization follows from this unicity together with the identification of $\mathfrak{q}_k\overline{P^0_{\mathbb{F}}}$ which is given in Proposition 3.4.

The kernel of the surjection $\overline{P^0_{\mathbb{F}}} \twoheadrightarrow \mathfrak{q}_k\overline{P^0_{\mathbb{F}}}$ has a filtration with subquotients of the form $\tilde{\Gamma}^{l(q-1)}$ with $l > k$. The functor $\tilde{\Gamma}^{l(q-1)}$ is zero when evaluated on spaces with $\dim V \leq k$, by Lemma 2.2 (3). It follows that the kernel is zero when evaluated on such spaces. Thus $\overline{P^0_{\mathbb{F}}}(V) \to \mathfrak{q}_k\overline{P^0_{\mathbb{F}}}(V)$ is an isomorphism when $\dim V \leq k$ and the result follows from the first part of the Lemma. □

4. The flag morphism of Crabb and Hubbuck

Throughout this section, let r denote a fixed positive integer and $\underline{s}$ denote a fixed decreasing sequence of positive integers, $s_1 \geq s_2 \geq \ldots \geq s_r > s_{r+1} = 0$.

Notation 4.1. For each positive integer s, let ϕ_s be the element of $\mathrm{Hom}_{\mathscr{F}}(\overline{P^0_{\mathbb{F}}}, \Gamma^{[s]_q})$ which sends the canonical generator of $\overline{P^0_{\mathbb{F}}}(\mathbb{F}) \cong \mathbb{F}[\mathfrak{Flag}_1](\mathbb{F})$ to $\iota^{\otimes [s]_q}$, where ι is any generator of $\mathbb{F}$. (The morphism is independent of the choice of ι).

Definition 4.2. For $\underline{s}$ a sequence of positive integers, let $\tilde{\phi}_{\underline{s}}$ denote the morphism

$$\tilde{\phi}_{\underline{s}} : P_{\mathbb{F}^r} \cong P_{\mathbb{F}}^{\otimes r} \xrightarrow{\otimes \phi_{s_i}} \bigotimes_{i=1}^{r} \Gamma^{[s_i]_q} \to \Gamma^{[\underline{s}]_q}$$

in which the second morphism is induced by the product.

The following Proposition is proved in the case $q = 2$ in [3].

Proposition 4.3. *The morphism $\tilde{\phi}_{\underline{s}}$ factorizes as*

$$P_{\mathbb{F}}^{\otimes r} \twoheadrightarrow \mathbb{F}[\mathfrak{Flag}_r] \xrightarrow{\phi_{\underline{s}}} \Gamma^{[\underline{s}]_q}.$$

Proof. Fixing a basis of $\mathbb{F}^r$, a canonical basis element of $P_{\mathbb{F}^r}(V)$ is an ordered sequence (v_i) of r elements of V. The morphism $\tilde{\phi}_{\underline{s}}$ sends this generator to $\prod_{i=1}^{r} v_i^{\otimes [s_i]_q} = \prod_{i=1}^{r} \prod_{j=0}^{s_i - 1} v_i^{\otimes (q-1)q^j}$.

Using this notation, define a natural surjection $P_{\mathbb{F}^r} \twoheadrightarrow \mathbb{F}[\mathfrak{Flag}_r]$ by sending (v_i) to the flag $\langle v_1 \rangle < \langle v_1, v_2 \rangle < \ldots < \langle v_1, \ldots, v_r \rangle$ if the elements are linearly independent and zero otherwise. The proposition asserts that $\tilde{\phi}_{\underline{s}}$ factorizes across this surjection.

The result follows as in the proof of [3, Lemma 3.1], by applying Lemma 2.7. □

5. The embedding theorem

The purpose of this section is to prove the main result of the paper, stated here as Theorem 5.17, which gives a criterion for

$$\phi_{\underline{s}}(V) : \mathbb{F}[\mathfrak{Flag}_r](V) \to \Gamma^{[\underline{s}]_q}(V)$$

to be a monomorphism, where r is a positive integer and $\underline{s} = s_1 > s_2 > \ldots > s_r > s_{r+1} = 0$ is a strictly decreasing sequence of integers.

Remark 5.1. The morphism $\phi_{\underline{s}}$ is clearly not a monomorphism of functors, since the functor $\Gamma^{[\underline{s}]_q}$ is finite whereas $\mathbb{F}[\mathfrak{Flag}_r]$ is highly infinite.

However, Lemma 3.5 (2) provides the key calculational input, which is restated as the following:

Lemma 5.2. *Let s be a natural number. The morphism $\phi_s : \mathbb{F}[\mathfrak{Flag}_1] \cong \overline{P^0_{\mathbb{F}}} \to \Gamma^{[s]_q}$ induces a monomorphism $\phi_s(V)$ if $[s]_q \geq (q-1)\dim V$.*

The theorem is proved by an induction using Lemma 5.2 to provide the inductive step. The strategy involves composing $\phi_{\underline{s}}$ with a morphism $\delta_{\underline{s}}$ (defined below) to give a morphism $\psi_{\underline{s}}$ which is amenable to induction. The key to setting up the induction is Lemma 5.6.

Write $[\underline{s}]_q = r[s_r]_q + \Sigma_{i=1}^{r-1}([s_i]_q - [s_r]_q)$; thus the coproduct gives a morphism $\Delta : \Gamma^{[\underline{s}]_q} \to \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma([s_i]_q - [s_r]_q)}$. For each i, $[s_i]_q - [s_r]_q = q^{s_r}[s_i - s_r]_q$, hence there is an iterated Verschiebung morphism $\mathcal{V}^{s_r} : \Gamma^{\Sigma([s_i]_q - [s_r]_q)} \twoheadrightarrow \Gamma^{\Sigma[s_i - s_r]_q}$.

Definition 5.3.

(1) Let $\delta_{\underline{s}} : \Gamma^{[\underline{s}]_q} \to \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{i=1}^{r-1}[s_i - s_r]_q}$ be the composite morphism

$$\Gamma^{[\underline{s}]_q} \xrightarrow{\Delta} \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma([s_i]_q - [s_r]_q)} \xrightarrow{1 \otimes \mathcal{V}^{s_r}} \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{i=1}^{r-1}[s_i - s_r]_q}.$$

(2) Let $\psi_{\underline{s}} : \mathbb{F}[\mathfrak{Flag}_r] \to \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1}[s_j - s_r]_q}$ be the composite morphism $\delta_{\underline{s}} \circ \phi_{\underline{s}}$.

The following elementary observation is recorded as a Lemma.

Lemma 5.4. *Let V be a finite-dimensional vector space. If the morphism $\psi_{\underline{s}}(V)$ is a monomorphism, then $\phi_{\underline{s}}(V)$ is a monomorphism.*

Notation 5.5. Let $\underline{s'}$ denote the sequence (of length $r - 1$) of positive integers $(s_1 - s_r > \ldots > s_{r-1} - s_r > 0)$.

The following Lemma is the key to the inductive proof, and relies upon the fact that the iterated Verschiebung is used in the definition of $\psi_{\underline{s}}$. Observe that the Crabb-Hubbuck morphism associated to the sequence of integers $(s_r, \ldots, s_r)$ of length r induces a morphism

$$\phi_{(s_r,\ldots,s_r)} : \mathbb{F}[\mathfrak{Flag}_r] \to \Gamma^{r[s_r]_q}.$$

Lemma 5.6. *The morphism $\psi_{\underline{s}}$ identifies with the composite morphism*

$$\begin{tikzcd}
\mathbb{F}[\mathfrak{Flag}_r] \arrow[r, "\mathrm{diag}"] & \mathbb{F}[\mathfrak{Flag}_r] \otimes \mathbb{F}[\mathfrak{Flag}_r] \arrow[r, "1\otimes\pi_{r,r-1}"] & \mathbb{F}[\mathfrak{Flag}_r] \otimes \mathbb{F}[\mathfrak{Flag}_{r-1}] \arrow[d, "\phi_{(s_r,\ldots,s_r)}\otimes\phi_{\underline{s}'}"] \\
& & \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1}[s'_j]_q}
\end{tikzcd}$$

Proof. We are required to prove that the following diagram is commutative.

$$\begin{tikzcd}
\mathbb{F}[\mathfrak{Flag}_r] \arrow[r, "\mathrm{diag}"] \arrow[d, "\phi_{\underline{s}}"'] & \mathbb{F}[\mathfrak{Flag}_r] \otimes \mathbb{F}[\mathfrak{Flag}_r] \arrow[r, "1\otimes\pi_{r,r-1}"] & \mathbb{F}[\mathfrak{Flag}_r] \otimes \mathbb{F}[\mathfrak{Flag}_{r-1}] \arrow[d, "\phi_{(s_r,\ldots,s_r)}\otimes\phi_{\underline{s}'}"] \\
\Gamma^{[\underline{s}]_q} \arrow[rr, "\delta_{\underline{s}}"'] & & \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1}[s'_j]_q}.
\end{tikzcd}$$

Choose a surjection $P_{\mathbb{F}^r} \twoheadrightarrow \mathbb{F}[\mathfrak{Flag}_r]$ as in the proof of Proposition 4.3; it is equivalent to prove the commutativity of the diagram obtained by composition, replacing the top left entry by $P_{\mathbb{F}^r}$. The analysis of the composite morphism

$$P_{\mathbb{F}^r} \to \Gamma^{[\underline{s}]_q} \to \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1}[s'_j]_q}. \tag{3}$$

around the bottom of the diagram can then be carried out as follows.

The definition of the morphism $\delta_{\underline{s}}$ and of the morphism $\phi_{\underline{s}}$ implies that this composite factors across

$$\bigotimes_{i=1}^{r} \Gamma^{[s_i]_q} \xrightarrow{\mu} \Gamma^{[\underline{s}]_q} \xrightarrow{\Delta} \Gamma^{r[s_r]_q} \otimes \Gamma^{q^{s_r}\Sigma_{j=1}^{r-1}[s'_j]_q}$$

where μ denotes the product on divided powers and Δ the diagonal.

The exponential algebra structure of Γ^* (essentially the fact that these functors take values in bicommutative Hopf algebras) implies that there is a commutative diagram

$$\begin{tikzcd}
\bigotimes_{i=1}^{r} P_{\mathbb{F}} \arrow[r, "\bigotimes_{i=1}^{r}\phi_{s_i}"] \arrow[rr, bend left, "\tilde{\phi}_{\underline{s}}"] & \bigotimes_{i=1}^{r} \Gamma^{[s_i]_q} \arrow[r, "\mu"] \arrow[d, "\bigoplus\Delta"'] & \Gamma^{[\underline{s}]_q} \arrow[d, "\Delta"] \arrow[dd, bend left=60, "\delta_{\underline{s}}"] \\
& \bigoplus\left\{ \bigotimes_{i=1}^{r} \left(\Gamma^{\alpha_i} \otimes \Gamma^{\beta_i}\right)\right\} \arrow[r, "\bigoplus(\mu\otimes\mu)"] \arrow[dr, dashed, "\xi"'] & \Gamma^{r[s_r]_q} \otimes \Gamma^{q^{s_r}\Sigma_{j=1}^{r-1}[s'_j]_q} \arrow[d, "1\otimes\mathcal{V}^{s_r}"] \\
& & \Gamma^{r[s_r]_q} \otimes \Gamma^{\Sigma_{j=1}^{r-1}[s'_j]_q}
\end{tikzcd}$$

where the sum is labelled over sequences of pairs of natural numbers (α_i, β_i) satisfying $\alpha_i + \beta_i = [s_i]_q$ for each i and $\Sigma\alpha_i = r[s_r]_q$.

Consider the composite morphism ξ in the diagram; Lemma 2.8 implies that the only components of this morphism which are non-trivial are those corresponding to sequences (α_i, β_i) for which $\beta_i = q^{s_r}\beta'_i$ for natural numbers β'_i. The condition $\alpha_i + q^{s_r}\beta'_i = [s_i]_q$ implies that α_i is non-zero; it follows that $\alpha_i \geq [s_r]_q$, for each

i, since $\alpha_i \equiv [s_r]_q \mod (q^{s_r})$. The condition $\Sigma_i \alpha_i = r[s_r]_q$ therefore implies that $\alpha_i = [s_r]_q$ for each i. It follows that ξ has only one non-zero component and a straightforward verification shows that the composite corresponds to the composite around the top of the diagram given in the statement of the Lemma. □

The inductive argument is simplified using the following:

Lemma 5.7. *Let V be a finite-dimensional vector space for which the morphism $\phi_{\underline{s}'}(V)$ is a monomorphism. Then the morphism $\psi_{\underline{s}}(V)$ is a monomorphism if and only if the composite morphism*

$$\mathbb{F}[\mathfrak{Flag}_r] \xrightarrow{\text{diag}} \mathbb{F}[\mathfrak{Flag}_r] \otimes \mathbb{F}[\mathfrak{Flag}_r] \xrightarrow{\phi_{s_r,\ldots,s_r} \otimes \pi_{r,r-1}} \Gamma^{r[s_r]_q} \otimes \mathbb{F}[\mathfrak{Flag}_{r-1}]$$

induces a monomorphism when evaluated upon V.

Proof. This follows from the identification of $\psi_{\underline{s}}$ which is given in Lemma 5.6. □

Using the fact that $\mathbb{F}[\mathfrak{Flag}_{r-1}](V)$ is generated by complete flags of length $r-1$, this allows the decomposition into components.

Notation 5.8. For Φ a complete flag of length $r-1$ in V, let

(1) $\langle \Phi \rangle \leq V$ denote the $(r-1)$-dimensional subspace of V defined by Φ;
(2) $\mathbb{F}[\mathfrak{Flag}_r]_\Phi(V)$ denote the subspace generated by flags containing Φ;
(3) γ_Φ denote the image of $[\Phi] \in \mathbb{F}[\mathfrak{Flag}_{r-1}](V)$ under the Crabb-Hubbuck morphism $\phi_{s_r,\ldots,s_r}(V) : \mathbb{F}[\mathfrak{Flag}_{r-1}](V) \to \Gamma^{(r-1)[s_r]_q}(V)$.

Remark 5.9. The space $\mathbb{F}[\mathfrak{Flag}_r]_\Phi(V)$ is isomorphic to $\mathbb{F}[\mathfrak{Flag}_1](V/\langle\Phi\rangle)$.

Lemma 5.10. *Let V be a finite-dimensional vector space for which the morphism $\phi_{\underline{s}'}(V)$ is a monomorphism. The morphism $\psi_{\underline{s}}(V)$ is a monomorphism if and only if, for each complete flag Φ in V of length $r-1$, the restriction of*

$$\mathbb{F}[\mathfrak{Flag}_r](V) \stackrel{\phi_{s_r,\ldots,s_r}}{\longrightarrow} \Gamma^{r[s_r]_q}(V)$$

to $\mathbb{F}[\mathfrak{Flag}_r]_\Phi(V)$ is a monomorphism.

Proof. A straightforward consequence of Lemma 5.7. □

Notation 5.11. Let V be a finite-dimensional vector space and Φ be a complete flag in V of length $r-1$. Let ρ_Φ denote the composite linear map

$$\mathbb{F}[\mathfrak{Flag}_r]_\Phi(V) \stackrel{\cong}{\to} \mathbb{F}[\mathfrak{Flag}_1](V/\langle\Phi\rangle) \stackrel{\phi_{s_r}}{\to} \Gamma^{[s_r]_q}(V/\langle\Phi\rangle)$$

induced by the projection $V \twoheadrightarrow V/\langle\Phi\rangle$ and the morphism ϕ_{s_r}.

Notation 5.12. For V a finite-dimensional vector space, Φ a complete flag in V of length $r-1$ and σ a section of the projection $V \twoheadrightarrow V/\langle\Phi\rangle$, let

$$\gamma_\Phi \cap_\sigma : \Gamma^{[s_r]_q}(V/\langle\Phi\rangle) \to \Gamma^{r[s_r]_q}(V)$$

denote the linear morphism induced by the section σ followed by the product with γ_Φ with respect to the algebra structure of $\Gamma^*(V)$.

Lemma 5.13. *Let V, Φ and σ be as above. The linear morphism*

$$\gamma_\Phi \cap_\sigma : \Gamma^{[s_r]_q}(V/\langle\Phi\rangle) \to \Gamma^{r[s_r]_q}(V)$$

is a monomorphism.

Proof. The result follows from the exponential structure of the divided power functors, since the element γ_Φ is the image of an element of $\Gamma^{(r-1)[s_r]_q}(\langle\Phi\rangle)$ under the morphism induced by the natural inclusion. □

Lemma 5.14. *Let V, Φ, σ be as above. The restriction of $\phi_{s_r,\ldots,s_r}(V)$ to $\mathbb{F}[\mathfrak{Flag}_r]_\Phi(V)$ identifies with the linear morphism $(\gamma_\Phi \cap_\sigma) \circ \rho_\Phi$.*

Proof. The result follows from the definition of the morphism $\phi_{s_r,\ldots,s_r}(V)$. $\square$

Lemmas 5.13 and 5.14 together imply the following result:

Lemma 5.15. *Let V be a finite-dimensional vector space and Φ be a complete flag in V of length $r-1$. The restriction of $\phi_{s_r,\ldots,s_r}(V)$ to $\mathbb{F}[\mathfrak{Flag}_r]_\Phi(V)$ is a monomorphism if and only if*

$$\phi_{s_r} : \mathbb{F}[\mathfrak{Flag}_1](V/\langle\Phi\rangle) \to \Gamma^{[s_r]_q}(V/\langle\Phi\rangle)$$

is a monomorphism.

Remark 5.16. By lemma 5.2, a sufficient condition is

$$[s_r]_q \geq (q-1)\dim(V/\langle\Phi\rangle) = (q-1)(\dim V - r + 1).$$

When $q = 2$, this is an equivalent condition.

Putting these results together, one obtains the following generalization of [3, Proposition 3.10].

Theorem 5.17. *Suppose that the sequence $\underline{s}$ satisfies the condition $[s_i - s_{i+1}]_q \geq (q-1)(\dim V - i + 1)$, for $1 \leq i \leq r$. Then the morphism $\phi_{\underline{s}}$ induces a monomorphism*

$$\mathbb{F}[\mathfrak{Flag}_r](V) \hookrightarrow \Gamma^{[\underline{s}]_q}(V).$$

Proof. The result is proved by induction upon r, starting with the initial case, $r = 1$, which is provided by Lemma 5.2. For the inductive step, by Lemma 5.4, it is sufficient to show that $\psi_{\underline{s}}(V)$ is a monomorphism, under the given hypotheses.

Observe that the hypotheses upon $\underline{s}$ imply that $\underline{s}'$ also satisfy the hypotheses with respect to V, so that the morphism $\phi_{\underline{s}'}(V)$ is injective, by induction. Hence Lemma 5.10 reduces the proof to showing that the restriction of $\phi_{s_r,\ldots,s_r}$ to $\mathbb{F}[\mathfrak{Flag}_r]_\Phi(V)$ is a monomorphism, for each complete flag Φ of length $r-1$ in V. The inductive step is completed by combining Lemma 5.15 with Lemma 5.2. $\square$

6. A stabilization result

The techniques employed in the proof of Theorem 5.17 can be used to provide further information on the nature of the embedding results. For instance, one has a direct proof of the following stabilization result.

Proposition 6.1. *Let $\underline{s}$ be a sequence of integers $(s_1 > \ldots > s_r > s_{r+1} = 0)$ and V be a finite-dimensional vector space for which the morphism $\phi_{\underline{s}}(V) : \mathbb{F}[\mathfrak{Flag}_r](V) \to \Gamma^{[\underline{s}]_q}(V)$ is a monomorphism.*

Let $\underline{s^+}$ denote the sequence given by $s_i^+ = s_i + 1$, for $1 \leq i \leq r$, and $s_{r+1}^+ = 0$. Then the morphism

$$\phi_{\underline{s^+}}(V) : \mathbb{F}[\mathfrak{Flag}_r](V) \to \Gamma^{[\underline{s^+}]_q}(V)$$

is a monomorphism.

Proof. The diagonal induces a morphism $\Gamma^{[\underline{s^+}]_q} \to \Gamma^{q[\underline{s}]_q} \otimes \Gamma^{(q-1)r}$. Hence, composing with the Verschiebung on the first morphism gives $\eta : \Gamma^{[\underline{s^+}]_q} \to \Gamma^{[\underline{s}]_q} \otimes \Gamma^{(q-1)r}$, as in Definition 5.3.

There is a commutative diagram

$$\begin{tikzcd}
\mathbb{F}[\mathfrak{Flag}_r] \arrow[r, "\phi_{\underline{s^+}}"] \arrow[d, "\mathrm{diag}"'] & \Gamma^{[\underline{s^+}]_q} \arrow[d, "\eta"] \\
\mathbb{F}[\mathfrak{Flag}_r] \otimes \mathbb{F}[\mathfrak{Flag}_r] \arrow[r, "\phi_{\underline{s}} \otimes \phi_{1,\ldots,1}"'] & \Gamma^{[\underline{s}]_q} \otimes \Gamma^{(q-1)r},
\end{tikzcd}$$

the commutativity of which is established by an argument similar to that employed in the proof of Lemma 5.6.

It suffices to show that the composite

$$\mathbb{F}[\mathfrak{Flag}_r](V) \xrightarrow{\phi_{\underline{s}^+}} \Gamma^{[\underline{s}^+]_q}(V) \xrightarrow{\eta} \Gamma^{[\underline{s}]_q} \otimes \Gamma^{(q-1)r}(V)$$

is a monomorphism. By hypothesis, the morphism $\phi_{\underline{s}}(V)$ is a monomorphism. As in the inductive step of the proof of Theorem 5.17, the result then follows from the fact that the morphism $\mathbb{F}[\mathfrak{Flag}_r](V) \to \Gamma^{(q-1)r}(V)$ is non-trivial. The latter follows from the fact that the hypothesis upon $\phi_{\underline{s}}(V)$ implies that V has dimension at least r. □

Remark 6.2. This argument is related to standard techniques using the Kameko Sq^0 operation [5], which is based in an essential way upon the analysis of the Verschiebung morphism.

References

[1] Mohamed Ali Alghamdi, M. C. Crabb, and J. R. Hubbuck, *Representations of the homology of BV and the Steenrod algebra. I*, Adams Memorial Symposium on Algebraic Topology, 2 (Manchester, 1990), London Math. Soc. Lecture Note Ser., vol. 176, Cambridge Univ. Press, Cambridge, 1992, pp. 217–234. MR MR1232208 (94i:55022)

[2] J. Michael Boardman, *Modular representations on the homology of powers of real projective space*, Algebraic topology (Oaxtepec, 1991), Contemp. Math., vol. 146, Amer. Math. Soc., Providence, RI, 1993, pp. 49–70. MR MR1224907 (95a:55041)

[3] M. C. Crabb and J. R. Hubbuck, *Representations of the homology of BV and the Steenrod algebra. II*, Algebraic topology: new trends in localization and periodicity (Sant Feliu de Guíxols, 1994), Progr. Math., vol. 136, Birkhäuser, Basel, 1996, pp. 143–154. MR MR1397726 (97h:55018)

[4] Vincent Franjou, Eric M. Friedlander, Alexander Scorichenko, and Andrei Suslin, *General linear and functor cohomology over finite fields*, Ann. of Math. (2) **150** (1999), no. 2, 663–728. MR MR1726705 (2001b:14076)

[5] Masaki Kamako, *Products of projective spaces as Steenrod modules*, Ph.D. thesis, Johns Hopkins University, 1990.

[6] Nicholas J. Kuhn, *Generic representations of the finite general linear groups and the Steenrod algebra. I*, Amer. J. Math. **116** (1994), no. 2, 327–360. MR MR1269607 (95c:55022)

[7] ———, *Generic representations of the finite general linear groups and the Steenrod algebra. II*, *K*-Theory **8** (1994), no. 4, 395–428. MR MR1300547 (95k:55038)

[8] J. Repka and P. Selick, *On the subalgebra of $H_*((\mathbf{R}\mathrm{P}^\infty)^n; F_2)$ annihilated by Steenrod operations*, J. Pure Appl. Algebra **127** (1998), no. 3, 273–288. MR MR1617199 (99d:55012)

[9] Grant Walker and Reg Wood, *Embedding the Steinberg and the flag representations of $GL(n, \mathbb{F}_q)$ in the divided power algebra*, Notes from talk at Larry Smith Emertierungsfeier, September 2007.

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